TECHNICAL NOTE

Boundary layer flow at a three-dimensional stagnation point in power-law non-Newtonian fluids

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Similarity solutions for the three-dimensional flow and heat transfer of a power-law fluid near a stagnation point of an isothermal surface are presented. The results of the numerical integrations are given in tables and shown on graphs for some different values of the power-law index n, geometric parameter c, and the Prandtl number Pr. Whenever possible, these results are compared with available analytical solutions and found to be highly accurate.

Keywords: boundary layers; stagnation-point flow; non-Newtonian fluids; rheology

Introduction

The problem of flow of a Newtonian fluid in the vicinity of a three-dimensional (3-D) stagnation point on a regular surface has been studied extensively in the literature (Banks 1967; Cooke and Robins 1970; Davey 1961; Gersten et al. 1978; Ghoshal and Ghoshal 1970; Hayday and Bowlus 1967; Howarth 1951; Kumari and Nath 1980; Libby 1976, 1977; Nath and Meena 1977; Reshotko 1958; Vimala and Nath 1975; Wadia 1985; Wortman 1971; Wortman and Mills 1971). These investigations were motivated by the basic nature of the boundary layer flow at such points, by the exact applicability there of similarity solutions, and by their relevance to the leading edge and nose regions of bodies in high-speed flight. The solution is of immense importance in the design of thermal protection systems for launch vehicles, as well as for spacecraft reentering planetary atmospheres at hypersonic speeds. There should also be mentioned the turbomachinery applications of the similarity solutions for the viscous flow in the vicinity of an axisymmetric stagnation point on a circular cylinder, with an oscillating main stream, see Gorla (1979, 1988a, 1988b).

The object of this article is to study the flow of an incompressible fluid obeying the Ostwald-de-Waele power-law model near a 3-D stagnation point of attachment on an isothermal regular surface. As pointed out by Gorla (1992a, 1992b) and Wangskarn et al. (1992), the power-law model is a relevant model for non-Newtonian fluids. The values of n used in this article represent physical applications. It is important to note that applications for such systems can be in the molten plastics, food, pulp, and paper and petrochemical industries. The spirit of the argument is the same as in the article by Hayday and Bowlus (1967); that is, the appropriate idealization of the flow is assumed to be that near a plane wall, with main-stream boundary conditions given by Equation 1, and a similarity solution of the inviscid form of the governing equations is sought. The derived ordinary differential equations are solved numerically for a wide range of three arbitrary parameters. Comparisons with other available solutions show excellent agreement.

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Governing equations

Consider a steady 3-D flow of an incompressible fluid obeying the power-law model in the vicinity of a stagnation point on a regular surface, which is held at a constant temperature T_w while the main-stream flow has the temperature $T_\infty(<T_w)$. A coordinate system (x, y, z) is introduced with the stagnation point in question at (0, 0, 0). The coordinate normal to the body is z, and x and y are in the directions of the two principal curvatures. In this coordinate system, the velocity of the main stream flow has components

$$U = ax,\ V = by,\ W = -(a + b)z \tag{1}$$

for some constants a and b. The signs and relative magnitude of a and b determine the nature of the stagnation point. In this article, we speak of a point of attachment if the normal component of the main stream velocity is directed toward the wall, that is, if $(a + b) > 0$. In the opposite case we speak of a point of separation. If a and b have the same sign, the stagnation point is termed a nodal point; otherwise it is a saddle point.

We assume that the following transport properties apply for the power-law viscosity model (Gryglaszewski and Saljnikov 1989; Pop and Gorla 1990, 1991; Shvets and Vishnevskiy 1987).

$$\tau_{ij} = -p\delta_{ij} + K|\tfrac{1}{2}I_2|^{(n-1)/2}e_{ij} \tag{2}$$

$$q = -k|\tfrac{1}{2}I_2|^{s}\,\text{grad}\,T \tag{3}$$

The boundary layer equations expressing the principles of conservation of mass, momentum, and energy in the vicinity of the stagnation point are

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \tag{4}$$

$$u\frac{\partial u}{\partial y} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} = U\frac{\partial U}{\partial x} + V\frac{\partial U}{\partial y} + \frac{K}{\rho}\frac{\partial}{\partial z}\left(\left|\frac{\partial u}{\partial z}\right|^{n-1}\frac{\partial u}{\partial z}\right) \tag{5}$$

$$u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} = U\frac{\partial V}{\partial x} + V\frac{\partial V}{\partial y} + \frac{K}{\rho}\frac{\rho}{\partial z}\left(\left|\frac{\partial u}{\partial z}\right|^{n-1}\frac{\partial v}{\partial z}\right) \tag{6}$$

$$u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} + w\frac{\partial T}{\partial z} = \alpha\frac{\partial}{\partial z}\left(\left|\frac{\partial u}{\partial z}\right|^{s}\frac{\partial T}{\partial z}\right) \tag{7}$$

where (u, v, w) are the velocity components along (x, y, z)-directions, ρ is the density, and α is the thermal diffusivity of the fluid. The boundary conditions of Equations 4 to 7 are

$$u(x, y, 0) = v(x, y, 0) = w(x, y, 0) = 0 \tag{8a}$$

$$T(x, y, 0) = T_w \tag{8b}$$

$$u(x, y, \infty) = U,\ v(x, y, \infty) = V,\ T(x, y, \infty) = T_\infty \tag{8c}$$

It can be easily shown that the above system consisting of Equations 4 to 7 subject to the boundary conditions (Equation 8) admits similarity solutions only if $S = n - 1$. In this article, we have not considered other values of S. We now present a compatible representation for u, v, w, and T in the following manner:

$$u = axf'(\eta),\ v = byg'(\eta) \tag{9a}$$

$$w = -a\left(\frac{a^{2-n}}{K/\rho}\right)^{-1/(1+n)}\left(\frac{2n}{1+n}f + \frac{1-n}{1+n}\eta f' + cg\right)x^{(n-1)/(n+1)} \tag{9b}$$

$$\theta(\eta) = (T - T_\infty)/(T_w - T_\infty) \tag{9c}$$

where

$$\eta = z\left(\frac{a^{2-n}}{K/\rho}\right)^{1/(n+1)}x^{(1-n)/(1+n)} \tag{10}$$

is a similarity variable, and prime denotes differentiation with respect to η. We mention at this point that $n = 1$ represents a Newtonian fluid, and $n < 1$ and $n > 1$ correspond to the cases of pseudoplastic and dilatant fluids, respectively.

Using the transformation defined by Equations 9 and 10, Equations 4 to 7 become

$$(|f''|^{n-1}f'')' + \left(\frac{2n}{n+1}f + cg\right)f'' = (f')^2 - 1 \tag{11}$$

$$(|f''|^{n-1}g'')' + \left(\frac{2n}{n+1}f + cg\right)g'' = c[(g')^2 - 1] \tag{12}$$

$$\frac{1}{\Pr}(|f''|^{n-1}\theta')' + \left(\frac{2n}{n+1}f + cg\right)\theta' = 0 \tag{13}$$

The problem is completely posed by adding the boundary conditions

$$f(0) = f'(0) = g(0) = g'(0) = 0,\ \theta(0) = 1 \tag{14a}$$

$$f'(\infty) = 1,\ g'(\infty) = 1,\ \theta(\infty) = 0 \tag{14b}$$

In the preceding equations $\Pr = K/(\rho\alpha)$ is the Prandtl number, and $c = b/a$ is a geometric parameter. It is important to note that for $c = 1$, then $f = g$. Equations 11 to 13 describe the flow of a power-law fluid near a stagnation point on a body of revolution. Then $c = 0$ corresponds to the two-dimensional flow. The dimensionless wall skin friction coefficients C_{fx} and C_{fy} as well as the Nusselt number Nu may be written as

$$\frac{C_{fx}\,\mathrm{Re}_x^{1/(n+1)}}{2} = |f''(0)|^{n-1}f''(0) \tag{15}$$

$$\frac{C_{fy}\,\mathrm{Re}_x^{1/(n+1)}}{2} = |f''(0)|^{n-1}g''(0) \tag{16}$$

$$\mathrm{Nu}\,\mathrm{Re}_x^{1/(n+1)} = -|f''(0)|^{n-1}\theta'(0) \tag{17}$$

where $\mathrm{Re}_x = (ax)^{2-n}x^n/(K/\rho)$ is a local Reynolds number based on the main-stream velocity U.

Notation

a, b	Constants
c	Geometric parameter
C_{fx}, C_{fy}	Skin friction coefficients
e_{ig}	Strain rate tensor
f, g	Similarity functions
I_2	Second invariant of the strain rate tensor
k	Thermal conductivity
K	Consistency index
n	Fluid power-law index
Nu	Nusselt number
p	Pressure
Pr	Prandtl number
q	Heat flux
Re_x	Local Reynolds number
s	Heat-transfer power-law index
T	Temperature
u, v, w	Velocity components in the boundary layer along (x, y, z)-directions
U, V, W	Velocity component in the main stream along (x, y, z)-directions
x, y, z	Local orthogonal co-ordinate system

Greek symbols

α	Thermal diffusivity
δ_{ij}	Unit tensor
η	Similarity variable
θ	Nondimensional temperature
ρ	Density
τ_{ij}	Stress tensor

Subscripts

w	Refers to surface values
∞	Refers to values in the main stream

Superscript

Differential with respect to η

Results and discussion

Equations 11 to 13 with the boundary conditions (Equation 14) constitute a nonlinear system of ordinary differential equations with three arbitrary parameters and define a difficult nonlinear two-point boundary value problem. We have not been able to find an exact analytical solution of this system. Hence, we solved these equations numerically. The asymptotic boundary conditions are satisfied at the edge of the boundary layer by adjusting the initial conditions so that the mean square error between the computed variables and the asymptotic values is minimized. Convergence to a solution is rapid and appears to be somewhat insensitive to the first guesses of the initial conditions. Converged numerical solutions were obtained for several values of the power-law index n and geometric parameter c in the ranges $0.5 \leq n \leq 2.0$ and $0 \leq c \leq 1$. The Prandtl number Pr has been taken 10 and 100, respectively.

Table 1 presents numerical values for $f''_{(0)}$, $g''_{(0)}$, and $\theta'_{(0)}$ for a range of values for n, c, and Pr. Using this tabulated data, one may easily compute C_{fx}, C_{fy} and Nu. To confirm the accuracy of our numerical procedure, we have compared our data for $n = 1$ (Newtonian fluid) with those reported by Hayday and Bowlus (1967); as expected, we find that our results are in excellent agreement with the literature data. For example, for $n = 1$, $c = 1$, and $Pr = 10$, our results for $f''_{(0)}$, $g''_{(0)}$, and $-\theta'_{(0)}$ are 1.314718, 1.314718, and 1.750868, whereas the values reported by Hayday and Bowlus (1967) are 1.315, 1.315, and 1.75, respectively.

Figures 1 and 2 display results for the friction factors C_{fx} and C_{fy} *versus* the geometric number c with n as a parameter. It is observed that the friction factor increases with c as well as n. Figures 3 and 4 show the results for the heat-transfer rate *versus* c for n ranging from 0.5 to 2.0. We observe that the Nusselt number increases with c as well as n. Tables 2 to 5 present numerical results of C_{fx}, C_{fy} and Nu for several values of the parameters n, c, and Pr. We see from these tables that for $n = 1$ (Newtonian fluid), the present results are in excellent agreement with the analytical solutions reported by Hayday and Bowlus (1967).

From the numerical results, it was observed that as c increases, the velocity distribution in the boundary layer becomes more uniform. As the value of n increases we observe that the velocity distribution tends to a more linear shape. The results for pseudoplastic fluids ($n < 1$) and dilatant fluids ($n > 1$) have not been reported in the literature so far. As the geometric parameter c decreases, the thermal boundary layer thickness increases, and the temperature distribution becomes

Table 1 Values of $f''(0)$, $g''(0)$ and $\theta'(0)$ for various values of c and n

n	c	$f''(0)$	$g''(0)$	$\theta'(0)$ Pr = 10	$\theta'(0)$ Pr = 100
0.5	0.00	1.059503	0.338657	−0.912135	−2.318206
	0.25	1.072028	0.579341	−1.009823	−2.509958
	0.50	1.090248	0.789032	−1.137215	−2.778642
	0.75	1.112155	0.973931	−1.274107	−3.075508
	1.00	1.136453	1.136453	−1.411911	−3.378229
0.8	0.00	1.176826	0.456216	−1.204040	−2.791029
	0.25	1.190752	0.711423	−1.293850	−2.971160
	0.50	1.209737	0.920305	−1.407481	−3.211250
	0.75	1.231693	1.098935	−1.528623	−3.472813
	1.00	1.255475	1.255475	−1.649526	−3.738806
1.0	0.00	1.232591	0.577523	−1.338799	−2.986337
	0.25	1.247174	0.802362	−1.425879	−3.163442
	0.50	1.265891	1.013584	−1.530611	−3.386156
	0.75	1.287053	1.171969	−1.640685	−3.625460
	1.00	1.314718	1.314718	−1.750868	−3.867898
1.5	0.00	1.319107	0.824933	−1.536622	−3.225955
	0.25	1.332718	0.985889	−1.616217	−3.392089
	0.50	1.348810	1.131860	−1.728217	−3.578016
	0.75	1.366772	1.264215	−1.790287	−3.870251
	1.00	1.386066	1.386066	−1.877603	−4.024238
2.0	0.00	1.356015	0.962891	−1.618329	−3.273704
	0.25	1.367773	1.088000	−1.689623	−3.424799
	0.50	1.381253	1.204114	−1.762705	−3.584469
	0.75	1.396222	1.311475	−1.835472	−3.746359
	1.00	1.412457	1.412457	−1.906609	−3.906501

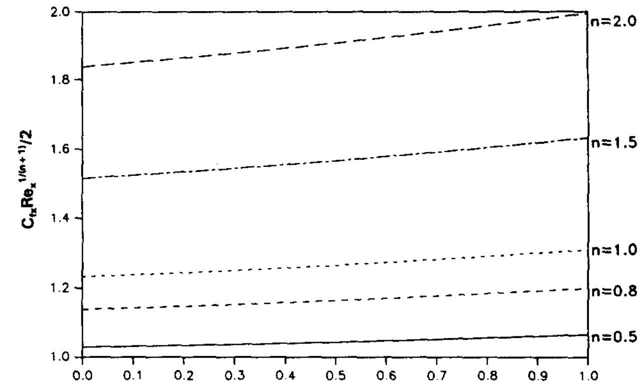

Figure 1 C_{fx} versus c

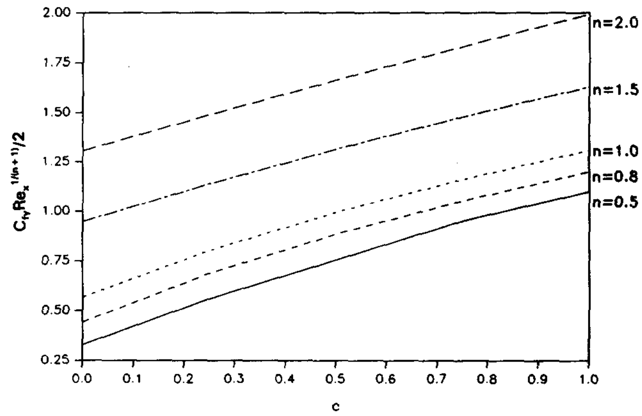

Figure 2 C_{fy} versus c

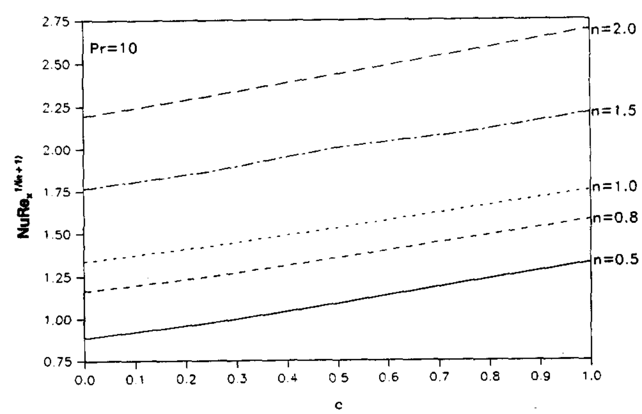

Figure 3 Nusselt number versus c for Pr = 10

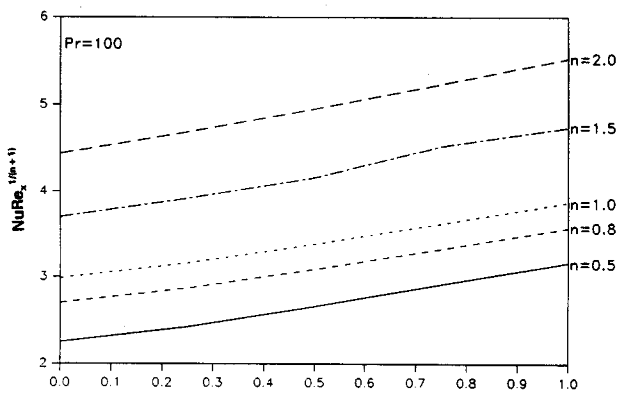

Figure 4 Nusselt number versus c for Pr = 100

Table 2 Variation of $C_{fx}Re_x^{1/(n+1)/2}$ with n and c

n	$c = 0.00$	$c = 0.25$	$c = 0.50$	$c = 0.75$	$c = 1.00$
0.5	1.029322	1.035388	1.044149	1.054588	1.066046
0.8	1.139121	1.149892	1.164536	1.181414	1.199628
1.0	1.232591	1.247174	1.265891	1.287053	1.314718
1.0*	1.227	1.245	1.265	1.289	1.315
1.5	1.515026	1.538535	1.566485	1.597880	1.631834
2.0	1.838777	1.870880	1.907860	1.949436	1.995035

* from Hayday and Boulos (1967)

Table 3 Variation of $C_{fy}Re_x^{1/(n+1)}/2$ with n and c

n	$c = 0.00$	$c = 0.25$	$c = 0.50$	$c = 0.75$	$c = 1.00$
0.5	0.329010	0.559540	0.755669	0.952736	1.069750
0.8	0.441599	0.687011	0.885917	1.054073	1.201218
1.0	0.577523	0.823621	1.013584	1.171969	1.314718
1.0*	0.585	0.838	1.014	1.172	1.315
1.5	0.947455	1.138144	1.314523	1.477981	1.630223
2.0	1.305695	1.488137	1.663186	1.831101	1.992371

* from Hayday and Boulos (1967)

Table 4 Variation of $NuReC_x^{1/(n+1)}$ with n and c for Pr = 10

n	$c = 0.00$	$c = 0.25$	$c = 0.50$	$c = 0.75$	$c = 1.00$
0.5	0.886152	0.975309	1.089130	1.208157	1.324437
0.8	1.165463	1.249453	1.354890	1.465875	1.576150
1.0	1.338800	1.425879	1.530611	1.640685	1.750868
1.0*	1.338792	1.427893	1.531235	1.641714	1.762036
1.5	1.764847	1.865816	2.007122	2.093010	2.210526
2.0	2.194478	2.311021	2.434742	2.562726	2.693003

* from Hayday and Boulos (1967)

more uniform. The thermal boundary layer thickness decreases as n increases.

Conclusion

In this article, we have presented similarity solutions for the momentum and energy equations governing a general 3-D flow of an incompressible power-law type of non-Newtonian fluid near a stagnation point. Results for the local friction factor and Nusselt number are presented for isothermal boundary conditions. The friction factor as well as heat-transfer rate increase with the geometric parameter c. Pseudoplastic fluids display less surface friction and heat-transfer rate when compared with dilatant fluids. Numerical results for the velocity and temperature distribution within the boundary layer are presented. The range of the flow behavior index n was from 0.5 to 2.0, whereas the geometric parameter c was allowed to vary from 0 to 1. The Prandtl number was assumed to be 10 and 100.

Table 5 Variation of $NuRe_x^{1/(n+1)}$ with n and c for Pr = 100

n	$c = 0.00$	$c = 0.25$	$c = 0.50$	$c = 0.75$	$c = 1.00$
0.5	2.252170	2.424173	2.661153	2.916314	3.168934
0.8	2.701604	2.869208	3.091262	3.331042	3.572493
1.0	2.986337	3.163442	3.386156	3.625460	3.867898
1.0*	2.986456	3.166771	3.387231	3.627456	3.870705
1.5	3.705088	3.915943	4.155447	4.524672	4.737788
2.0	4.439192	4.684348	4.951059	5.230750	5.517765

* from Hayday and Boulos (1967)

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